ISOTOPY CLASSES OF IMBEDDINGS

BY

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1. **Introduction.** The deleted product space X^* of a space X is $X \times X - \Delta$. If X is a finite polyhedron, let

$$P(X^*) = \bigcup \{\sigma \times \tau \mid \sigma \text{ and } \tau \text{ are simplexes of } X \text{ and } \sigma \cap \tau = \varnothing\}.$$

Hu [3] has shown that X^* and $P(X^*)$ are homotopically equivalent. In [4], the author has shown that if Y is a triod, then $P(Y^*)$ is a circle, and that up to homeomorphism the triod is the only tree (finite, contractible, 1-dimensional polyhedron) with this property. It is also shown in [4] that if X is a tree, then $H_1(X^*, Z)$ where Z is the integers, is a free abelian group. T. R. Brahana suggested to the author that if X is a tree, then there might be a connection between the number of generators of $H_1(X^*, Z)$ and the number of isotopy classes of imbeddings of the triod in X and that we might be able to extend this to higher dimensions.

In §2, we obtain a formula for computing the number of isotopy classes of imbeddings of the triod in a tree and show that there is a definite relation between this number and the 1-dimensional Betti number of the deleted product of the tree.

We show that up to homeomorphism there are at least two finite, contractible, 2-dimensional polyhedra, C and θ, which have the property that $P(C^*)$ and $P(\theta^*)$ are homeomorphic to the 2-sphere. There is at least one more finite, contractible, 2-dimensional polyhedron Λ whose deleted product has the homotopy type of the 2-sphere. However C can be imbedded in both θ and Λ, and in §4, we prove a collection of theorems which give a combinatorial method for computing the number of isotopy classes of imbeddings of C in a finite, contractible, 2-dimensional polyhedron.

The connection between the number of isotopy classes of imbeddings of C in a finite, contractible, 2-dimensional polyhedron X and the 2-dimensional Betti number of the deleted product of X is to be investigated in a forthcoming paper.

2. **Imbedding the triod.** Throughout this section, let Y denote a triod, let y_0 denote the vertex of Y of order 3, and let X denote a tree which is not an arc.

Gottlieb [2] defined a branch point as follows: Let S be a pathwise connected space. A point x of S is a *branch point* of S if $S - \{x\}$ has at least three path-components.

THEOREM 1. *If f: $Y \to X$ is an imbedding, then $f(y_0)$ is a branch point of X and $f(Y - \{y_0\})$ intersects exactly three path-components of $X - \{f(y_0)\}$.*

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Proof. Let P_1, P_2, and P_3 denote the three path-components of $Y-\{y_0\}$, and let Q_i $(i=1, 2, 3)$ denote the path-component of $X-\{f(y_0)\}$ which contains $f(P_i)$. Suppose $Q_i=Q_j$ for some $i\neq j$. Let $p_i\in f(P_i)$ and $p_j\in f(P_j)$. Then there exists a homeomorphism $h\colon I\to X-\{f(y_0)\}$ such that $h(0)=p_i$ and $h(1)=p_j$. Now there exists a homeomorphism $g\colon I\to Y$ such that $g(0)=f^{-1}(p_i)$ and $g(1)=f^{-1}(p_j)$. Since $g(t)=y_0$ for some $t\in I$, $f(g(I))\cup h(I)$ contains a simple closed curve. This contradicts the fact that X is a tree.

THEOREM 2. *If $f\colon Y\to X$ is an imbedding and $H\colon Y\times I\to X$ is an isotopy such that $H(y, 0)=f(y)$ for all $y\in Y$, then $H(y_0, t)=f(y_0)$ for all $t\in I$.*

Proof. Define a path $\sigma\colon I\to X$ by $\sigma(t)=H(y_0, t)$. Suppose there exists $t\in I$ such that $\sigma(t)\neq f(y_0)$. Then there exists $t_1\in I$ such that $\sigma(t_1)$ is not a vertex of X. Thus $H(y_0, t_1)=\sigma(t_1)$ is not a branch point of X. But $h_{t_1}\colon Y\to X$ defined by $h_{t_1}(y)=H(y, t_1)$ is an imbedding. Therefore, by Theorem 1, $h_{t_1}(y_0)$ is a branch point.

THEOREM 3. *If $f, g\colon Y\to X$ are imbeddings, then f is isotopic to g if and only if $f(y_0)=g(y_0)$ and, for each path-component P of $Y-\{y_0\}$, $f(P)$ and $g(P)$ are contained in the same path-component of $X-\{f(y_0)\}$.*

Proof. Suppose there exists an isotopy $H\colon Y\times I\to X$ such that $H(y, 0)=f(y)$ and $H(y, 1)=g(y)$ for each $y\in Y$. Then $f(y_0)=g(y_0)$ by Theorem 2. Suppose there exist a path-component P of $Y-\{y_0\}$ and path-components Q_1 and Q_2 of $X-\{f(y_0)\}$ such that $Q_1\neq Q_2$, $f(P)\subset Q_1$, and $g(P)\subset Q_2$. Let $y_1\in P$. Define a path $\sigma\colon I\to X$ by $\sigma(t)=H(y_1, t)$. Then $\sigma(0)=f(y_1)\in Q_1$ and $\sigma(1)=g(y_1)\in Q_2$. Since Q_1 and Q_2 are path-components of $X-\{f(y_0)\}$ and $Q_1\neq Q_2$, $\sigma(t_1)=f(y_0)$ for some $t_1\in I$. Now $h_{t_1}\colon Y\to X$ defined by $h_{t_1}(y)=H(y, t_1)$ is an imbedding. But $h_{t_1}(y_1)=H(y_1, t_1)=\sigma(t_1)=f(y_0)$, and $h_{t_1}(y_0)=H(y_0, t_1)=f(y_0)$. Thus we have a contradiction.

If $f(y_0)=g(y_0)$, and, for each path-component P of $Y-\{y_0\}$, $f(P)$ and $g(P)$ are contained in the same path-component of $X-\{f(y_0)\}$, then it is clear that f is isotopic to g.

THEOREM 4. *If m is the order of a vertex of maximum order in X, and, for each $j=3, 4, \ldots, m$, p_j is the number of vertices of order j, then the number of isotopy classes of imbeddings of Y in X is $\sum_{j=3}^{m} j(j-1)(j-2)p_j$.*

Proof. Let p be the number of vertices of X of order $\geqq 3$, and for each $i=1, 2, \ldots, p$, let n_i be the order of the ith vertex. Then it follows from Theorems 1, 2, and 3 that the number of isotopy classes of imbeddings of Y in X is $\sum_{i=1}^{p} n_i(n_i-1)(n_i-2)$. But $\sum_{i=1}^{p} n_i(n_i-1)(n_i-2)=\sum_{j=3}^{m} j(j-1)(j-2)p_j$.

THEOREM 5. *If m is the order of a vertex of maximum order in X, and, for each $j=3, 4, \ldots, m$, p_j is the number of vertices of order j, then $H_1(X^*, Z)$ is the free abelian group on $\sum_{j=3}^{m} [(j-2)(j-1)p_j]-1$ generators.*

Proof. Let p be the number of vertices of X of order $\geqq 3$, and for each $i=1, 2, \ldots, p$, let n_i be the order of the ith vertex. By Theorem 3.4 of [5], $H_1(X^*, Z)$ is the free abelian group on $\sum_{i=1}^{p} [(n_i-1)^2-(n_i-1)]-1$ generators. But

$$\sum_{i=1}^{p} [(n_i-1)^2-(n_i-1)]-1 = \sum_{j=3}^{m} [(j-1)(j-2)p_j]-1.$$

Thus, by comparing the formulas in Theorems 4 and 5, we see that there is a definite relation between the number of isotopy classes of imbeddings of Y in X and the 1-dimensional homology group of the deleted product of X.

3. **The 2-dimensional analog of the triod.** For each $i=1, 2, 3$, let σ_i be a 2-simplex, and let r be a 1-simplex. Throughout the remainder of this paper, let C denote the polyhedron, consisting of these simplexes and their faces, which satisfies the following conditions:

(1) r is not a face of σ_i for any i,

(2) there is a vertex c_0 which is a vertex of r and of σ_i for each i,

(3) for each $i<j$, $\sigma_i \cap \sigma_j$ is a 1-simplex r_{ij}, and

(4) $r_{ij} \neq r_{km}$ unless $i=k$ and $j=m$.

The polyhedron C is a cone with a "sticker" attached to its vertex c_0. We will continue to let r denote the 1-simplex described above. Also we will denote $C-r \cup \{c_0\}$ by D. Note that D is a disk.

Theorem 6. *The polyhedron $P(C^*)$ is homeomorphic to the 2-sphere.*

Proof. For each $i=1, 2, 3$, let r_i denote the 1-face of σ_i which does not have c_0 as a vertex. For each $i<j$, let c_{ij} denote the other vertex of r_{ij}, and let c denote the other vertex of r. The polyhedron $P(C^*)$ consists of the following 2-cells and their faces:

$\sigma_1 \times c_{23}$	$c_{12} \times \sigma_3$	$r_{12} \times r_3$	$r_2 \times r$
$\sigma_1 \times c$	$c_{13} \times \sigma_2$	$r_{13} \times r_2$	$r_3 \times r_{12}$
$\sigma_2 \times c_{13}$	$c_{23} \times \sigma_1$	$r_{23} \times r_1$	$r_3 \times r$
$\sigma_2 \times c$	$c \times \sigma_1$	$r_1 \times r_{23}$	$r \times r_1$
$\sigma_3 \times c_{12}$	$c \times \sigma_2$	$r_1 \times r$	$r \times r_2$
$\sigma_3 \times c$	$c \times \sigma_3$	$r_2 \times r_{13}$	$r \times r_3$

The proof now consists of only routine verifications, and hence it is omitted.

For each $i=1, 2, 3$, let τ_i be a 2-simplex, and suppose there is a 1-simplex s which is a face of τ_i for each i. Let u and v denote the vertices of s, and for each i, let u_i denote the vertex of τ_i which is different from u and v. Also for each i, denote the 1-faces of τ_i different from s by s_{i1} and s_{i2} so that $s_{i1} \cap s_{j1} \neq \varnothing \neq s_{i2} \cap s_{j2}$ but $s_{i1} \cap s_{j2} = \varnothing$ for $i \neq j$. Let θ denote the polyhedron consisting of these simplexes.

Theorem 7. *The polyhedron $P(\theta^*)$ is homeomorphic to the 2-sphere.*

Proof. The polyhedron $P(\theta^*)$ consists of the following 2-cells and their faces:

$\tau_1 \times u_2$	$u_1 \times \tau_2$	$s_{11} \times s_{22}$	$s_{22} \times s_{11}$
$\tau_1 \times u_3$	$u_1 \times \tau_3$	$s_{11} \times s_{32}$	$s_{22} \times s_{31}$
$\tau_2 \times u_1$	$u_2 \times \tau_1$	$s_{12} \times s_{21}$	$s_{31} \times s_{12}$
$\tau_2 \times u_3$	$u_2 \times \tau_3$	$s_{12} \times s_{31}$	$s_{31} \times s_{22}$
$\tau_3 \times u_1$	$u_3 \times \tau_1$	$s_{21} \times s_{12}$	$s_{32} \times s_{11}$
$\tau_3 \times u_2$	$u_3 \times \tau_2$	$s_{21} \times s_{32}$	$s_{32} \times s_{21}$

Again the proof now consists of only routine verifications.

Suppose we add a 2-simplex σ_4 to the polyhedron C so that r and r_{12} are faces of σ_4. Let r_4 denote the remaining 1-face of σ_4, and let Λ denote the polyhedron obtained in this manner.

THEOREM 8. *The polyhedron $P(\Lambda^*)$ has the homotopy type of the 2-sphere.*

Proof. The polyhedron $P(\Lambda^*)$ consists of the following cells and their faces:

$\sigma_3 \times r_4$	$\sigma_1 \times c$	$c \times \sigma_1$	$r_1 \times r$
$\sigma_4 \times r_3$	$\sigma_2 \times c_{13}$	$c \times \sigma_2$	$r_2 \times r_{13}$
$r_4 \times \sigma_3$	$\sigma_2 \times c$	$r_{13} \times r_2$	$r_2 \times r$
$r_3 \times \sigma_4$	$c_{13} \times \sigma_2$	$r_{23} \times r_1$	$r \times r_1$
$\sigma_1 \times c_{23}$	$c_{23} \times \sigma_1$	$r_1 \times r_{23}$	$r \times r_2$

It is now a routine matter to verify that $P(\Lambda^*)$ is homotopically equivalent to the 2-sphere.

It is clear that C can be imbedded in both θ and Λ.

4. **Imbedding** C. Throughout this section, let X denote a finite, contractible, 2-dimensional polyhedron.

DEFINITION 1. A point $x \in X$ is called a *c-point* of X if there exist 2-simplexes, $\tau_1, \tau_2, \ldots, \tau_n$, of X and a simplex τ of X such that:

(1) τ is not a face of τ_i for any i,

(2) x is a vertex of τ and of τ_i for each i,

(3) $\tau_n \cap \tau_1$ is a 1-simplex s_n,

(4) for each $i=1, 2, \ldots, n-1$, $\tau_i \cap \tau_{i+1}$ is a 1-simplex s_i, and

(5) $\tau_i \cap \tau_j = x$ unless i and j satisfy the conditions of either (3) or (4).

Note. By a collection of 2-simplexes satisfying Definition 1, we mean the 2-simplexes $\tau_1, \tau_2, \ldots, \tau_n$, i.e. we do not include τ even though it may be a 2-simplex.

THEOREM 9. *If f: $C \to X$ is an imbedding, then $f(c_0)$ is either a c-point of X or an interior point of a* 1*-simplex which is a face of at least three* 2*-simplexes.*

Proof. First suppose $f(c_0)$ is not a vertex of X. Then $f(c_0)$ is an interior point of either a 1-simplex or a 2-simplex. Since the interior of C is not homeomorphic to a subset of an open disk, it is easy to see that $f(c_0)$ cannot be either an interior point of a 2-simplex or an interior point of a 1-simplex which is a face of less than three 2-simplexes.

Now suppose $f(c_0)$ is a vertex of X. Since f is an imbedding, there is an arbitrarily small neighborhood U of $f(c_0)$ such that U contains a subset which is homeomorphic to C. Therefore $f(c_0)$ is a c-point.

Notation. If t is a point of a 1-simplex $s=\langle x, y\rangle$, then there exists a number λ such that $0\leqq\lambda\leqq 1$ and $t=\lambda x+(1-\lambda)y$. Let $[x, t]=\{z=\mu x+(1-\mu)y \mid \lambda\leqq\mu\leqq 1\}$.

Notation. If P is a locally finite polyhedron and v is a vertex of P, let $\mathrm{St}(v, P)$ denote the open star of v in P.

THEOREM 10. *If $f\colon C\to X$ is an imbedding and $f(c_0)$ is a c-point of X, then there exists a unique collection C_f of 2-simplexes of X satisfying Definition 1 such that (1) $f(C-r)$ intersects the interior of every simplex in C_f and (2) there exists a neighborhood U of $f(c_0)$ such that if τ is a simplex which is not a face of a simplex of C_f, then $f(C-r)\cap \mathrm{int}(\tau)\cap U=\varnothing$. Moreover there is a point t_f in r ($t_f\neq c_0$) such that $f([c_0, t_f])\cap\bigcup\{\tau_i \mid \tau_i\in C_f\}=\{f(c_0)\}$.*

Proof. Since f is continuous, there is a neighborhood V of c_0 such that $f(V)\subset \mathrm{St}(f(c_0), X)$. Let C' be a subset of V which is homeomorphic to C, and let

$$\Gamma = \{\tau \mid \tau \text{ is a 2-simplex of } X \text{ and } \mathrm{int}(\tau)\cap f(C'-r)\neq\varnothing\}.$$

Since $f(C')$ is homeomorphic to C and $f(c_0)$ is a vertex, Γ contains a collection C_f of 2-simplexes satisfying Definition 1. Suppose there exists a 2-simplex $\tau\in\Gamma-C_f$. Let $\Gamma'=\{\tau \mid \tau\in\Gamma-C_f\}$. If $\bigcup\{\tau \mid \tau\in\Gamma'\}\cap\bigcup\{\tau \mid \tau\in C_f\}=\{f(c_0)\}$, then $f(C'-r)$ is not connected. Therefore there exist 1-simplexes $s_1, s_2, \ldots, s_p$ such that $f(c_0)$ is a vertex of s_k for each k and $\bigcup\{\tau \mid \tau\in\Gamma'\}\cap\bigcup\{\tau \mid \tau\in C_f\}=\bigcup_{k=1}^{p} s_k$. Since $f(C'-r)\subset\bigcup\{\tau \mid \tau\in\Gamma\}$, $f(C'-r)-\bigcup_{k=1}^{p} s_k$ is not connected. Therefore

$$(C'-r)-f^{-1}\left(\bigcup_{k=1}^{p} s_k\right)$$

is not connected, and hence $p>1$. Let $K_1, K_2, \ldots, K_n$ be the components of $(C'-r)-f^{-1}(\bigcup_{k=1}^{p} s_k)$, and suppose $K_1, K_2, \ldots, K_n$ are ordered so that K_i and K_{i+1} have a common limit point different from c_0, K_n and K_1 have a common limit point different from c_0, and $f(K_1)\subset\bigcup\{\tau \mid \tau\in C_f\}$. Note that no three of the K_i's can have a common limit point different from c_0. Without loss of generality, we may assume that $f(K_2)\subset\bigcup\{\tau \mid \tau\in\Gamma'\}$. Let p_1 be a common limit point of K_1 and K_2 such that $p_1\in C'$ and $p_1\neq c_0$. There exists j ($1\leqq j\leqq p$) such that $f(p_1)\in s_j-\{f(c_0)\}$. Since C_f satisfies Definition 1, there exists i ($3\leqq i\leqq n$) such that $f(K_i)\subset\bigcup\{\tau \mid \tau\in C_f\}$ and $f(K_1)$ and $f(K_i)$ have a common limit point q_1 in $s_j-\{f(c_0)\}$. Thus $[f(c_0), q_1]\cap[f(c_0), f(p_1)]$ contains a point x different from $f(c_0)$. But $x\in[f(c_0), q_1]\cap[f(c_0), f(p_1)]$ and $x\neq f(c_0)$ implies that $f^{-1}(x)$ is a limit point of K_1, K_2, and K_i which is different from c_0. Therefore $\Gamma=C_f$.

Now let W be a neighborhood of c_0 such that $W\subset C'$, let U' be a neighborhood of $f(c_0)$ which does not intersect $f(C-W)$, and let $U=U'\cap\mathrm{St}(f(c_0), X)$. Let τ be any simplex which is not a face of a simplex of C_f. Then $f(C'-r)\cap\mathrm{int}(\tau)=\varnothing$.

Since $W \subset C'$, $f(C-C') \subset f(C-W)$. Let $x \in f(C-r)$. Then either $x \in f(C'-r)$ or $x \in f(C-C')$. If $x \in f(C'-r)$, then $x \notin \text{int}(\tau)$. If $x \in f(C-C')$, then $x \notin U$. Therefore $f(C-r) \cap \text{int}(\tau) \cap U = \varnothing$.

Suppose there is another collection C'_f of 2-simplexes of X satisfying Definition 1 and conditions (1) and (2) of the theorem. Then either there is a 2-simplex in C_f which is not in C'_f or there is a 2-simplex in C'_f which is not in C_f. Suppose τ is in C_f but not in C'_f. Then, since τ is not in C'_f, there is a neighborhood U of $f(c_0)$ such that $f(C-r) \cap U \cap \text{int}(\tau) = \varnothing$. But this contradicts the fact that τ is in C_f.

If $f(r) \cap \bigcup \{\tau \mid \tau \in C_f\} = \{f(c_0)\}$, then we can take t_f to be any point of r other than c_0. Suppose $f(r) \cap \bigcup \{\tau \mid \tau \in C_f\} \neq \{f(c_0)\}$. Let c be the other vertex of r, let

$$A = \{x = \mu c_0 + (1-\mu)c \mid 0 \leqq \mu < 1 \text{ and } f(x) \in \bigcup \{\tau \mid \tau \in C_f\}\},$$

and let $\lambda = \text{lub}\{\mu \mid \mu c_0 + (1-\mu)c \in A\}$. Suppose $\lambda = 1$. There exists a neighborhood U of $f(c_0)$ such that each point of $U \cap \bigcup \{\tau \mid \tau \in C_f\}$ is the image of a point of $(C-r) \cup \{c_0\}$ under f. Since f is continuous, there exists a neighborhood V of c_0 such that $f(V) \subset U$. Since $\lambda = 1$, there is a point $c' \in r \cap V$ such that $c' \neq c_0$ and $f(c') \in \bigcup \{\tau \mid \tau \in C_f\}$. Thus $f(c') \in \bigcup \{\tau \mid \tau \in C_f\} \cap U$, and hence f is not one-to-one. Therefore $\lambda < 1$. Let λ' be a number such that $\lambda < \lambda' < 1$, and let

$$t_f = \lambda' c_0 + (1-\lambda')c.$$

Note. If $f\colon C \to X$ is an imbedding such that $f(c_0)$ is a c-point of X, then, throughout this paper, we shall denote by t_f a point in r which has been chosen so that $t_f \neq c_0$ and $f([c_0, t_f]) \cap \bigcup \{\tau \mid \tau \in C_f\} = \{f(c_0)\}$.

DEFINITION 2. If τ and τ' are simplexes, then we say that τ and τ' are *joined by a chain of 2-simplexes* if there exists a sequence $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes such that:

(1) $\tau \cap \tau_1$ is a 1-simplex,

(2) $\tau_n \cap \tau'$ is a 1-simplex, and

(3) for each i, $\tau_i \cap \tau_{i+1}$ is a 1-simplex.

We say that $\tau_1, \tau_2, \ldots, \tau_n$ is a *chain of 2-simplexes joining* τ and τ'.

DEFINITION 3. If $f\colon C \to X$ is an imbedding such that $f(c_0)$ is a c-point of X, let

$$A_f = \{\tau \mid \tau \text{ is a simplex of } X, f(c_0) \text{ is a vertex of } \tau, \text{ either}$$

$$f([c_0, t_f]) \cap (\tau - \{f(c_0)\}) \neq \varnothing$$

$$\text{or } \tau \text{ is a face of a simplex } s \text{ with the property that}$$

$$f([c_0, t_f]) \cap (s - \{f(c_0)\}) \neq \varnothing,$$

$$\text{and } \tau \text{ is not a face of a simplex of } C_f\}.$$

If A_f contains a 1-simplex, choose a 1-simplex s_f in A_f. If A_f does not contain a 1-simplex, choose a 2-simplex τ in A_f and let s_f denote the line segment in τ from $f(c_0)$ to the barycenter of the 1-face of τ opposite $f(c_0)$. In either case the line segment s_f is called a *c-line*.

Note. We extend Definition 2 in the obvious way so that we can talk about a chain of 2-simplexes joining either two c-lines or a c-line and a simplex.

THEOREM 11. *Let $f, g\colon C \to X$ be imbeddings such that $f(c_0)=g(c_0)$ is a c-point of X. If f is isotopic to g under an isotopy H such that $H(c_0, t)=f(c_0)$ for each $t \in I$, then $C_f=C_g$ and either $s_f=s_g$ or there exists a chain $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes joining s_f and s_g such that $f(c_0)$ is a vertex of τ_i for each i and $\tau_i \cap \tau_{i+1}$ is not a face of a simplex of C_f for any i.*

Proof. Let $H\colon C\times I \to X$ be an isotopy such that $H(w, 0)=f(w)$ and $H(w, 1)=g(w)$ for each $w \in C$ and $H(c_0, t)=f(c_0)$ for each $t \in I$. For each $t \in I$, let $h_t\colon C \to X$ be the imbedding defined by $h_t(w)=H(w, t)$.

Suppose $C_f \neq C_g$, and let $t'=\text{lub}\{t \mid C_{h_t}=C_f\}$. Suppose $C_{h_{t'}}=C_f$. Let $\{t_i\}_{i=1}^{\infty}$ be a sequence of points such that $t_1<1$, $t_i>t'$ for each i, $t_i>t_{i+1}$ for each i, and $\lim_{i\to\infty} t_i=t'$. For each i, there exists a 2-simplex η_i of C_f such that $\eta_i \notin C_{h_{t_i}}$. Since C_f has only a finite number of simplexes, there is a 2-simplex η such that $\eta_i=\eta$ for an infinite number of i's. Let V' be a neighborhood of (c_0, t') such that $H(V')\subset \text{St}(f(c_0), X)$. There exists a connected neighborhood M' of c_0 and a neighborhood N' of t' such that $M'\times N'\subset V'$. Let $c_1 \in M' \cap D$ such that $H(c_1, t') \in \text{int}(\eta)$. Let V be any neighborhood of (c_1, t'). There exists a neighborhood M of c_1 and a connected neighborhood N of t' such that

$$M\times N \subset V\cap (M'\times N').$$

There exists i such that $t_i \in N$ and $\eta_i=\eta$. Since

$$M'\times\{t_i\} \subset V',\ H(M'\times\{t_i\}) \subset \text{St}(f(c_0), X).$$

Therefore, since $M'\times\{t_i\}$ is connected, $c_0 \in M'$, and $c_1 \in M' \cap D$, $H(c_1, t_i) \in C_{h_{t_i}}$. Therefore $H(V)\not\subset \text{int}(\eta)$, and hence H is not continuous. If $C_{h_{t'}}\neq C_f$, then $t'>0$, and, using essentially the same argument, we can show that H is not continuous. Therefore $C_f=C_g$.

Suppose $s_f\neq s_g$. For each $t \in I$, there exists a neighborhood V_t of c_0 and a neighborhood W_t of t such that $H(V_t\times W_t)\subset \text{St}(f(c_0), X)$. Let $V_{t_1}\times W_{t_1}, V_{t_2}\times W_{t_2}, \ldots, V_{t_n}\times W_{t_n}$ be a finite subcollection of $\{V_t\times W_t \mid t \in I\}$ which covers $\{c_0\}\times I$. Let $V=\bigcap_{i=1}^{n} V_{t_i}$. There V is a neighborhood of c_0 and $H(V\times I)\subset \text{St}(f(c_0), X)$. Let $t_H \in r(t_H\neq c_0)$ such that $[c_0, t_H]\subset V$, and let $c_1 \in [c_0, t_H] \cap [c_0, t_f] \cap [c_0, t_g]$ such that $c_1\neq c_0$.

We assert that there exists a neighborhood N of 1 such that if $t \in N$, then t_{h_t} can be chosen so that

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup\{\tau \mid \tau \in A_g\}-\{f(c_0)\} \neq \varnothing.$$

First suppose

$$g(\partial(C-r \cup \{f(c_0)\})) \cap \bigcup\{\tau \mid \tau \in C_g\} = \varnothing.$$

For each $x \in \partial(C-r \cup \{f(c_0)\})$, there exists a neighborhood M_x of x and a neighborhood N_x of 1 such that $H(M_x \times N_x) \cap \bigcup \{\tau \mid \tau \in C_g\} = \varnothing$. Let $M_{x_1} \times N_{x_1}$, $M_{x_2} \times N_{x_2}, \ldots, M_{x_n} \times N_{x_n}$ be a finite subcollection of

$$\{M_x \times N_x \mid x \in \partial(C-r \cup \{f(c_0)\})$$

which covers $\partial(C-r \cup \{f(c_0)\}) \times \{1\}$, and let $N' = \bigcap_{i=1}^m N_{x_i}$. Then N' is a neighborhood of 1, and $H[\partial(C-r \cup \{f(c_0)\}) \times N'] \cap \bigcup \{\tau \mid \tau \in C_g\} = \varnothing$. If $t \in N'$, then each point of $\bigcup \{\tau \mid \tau \in C_g\}$ is the image under h_t of some point of $C-r \cup \{f(c_0)\}$. Therefore $h_t([c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in C_g\} - \{f(c_0)\} = \varnothing$ if $t \in N'$. Thus we may assume that for $t \in N'$, $[c_0, c_1] \subset [c_0, t_{h_t}]$. Let $B_g = \{\tau \mid f(c_0)$ is a vertex of τ and $g([c_0, t_g]) \cap (\tau - \{f(c_0)\}) \neq \varnothing\}$. Then $W = \bigcup \{\text{int}(\tau) \mid \tau \in B_g\}$ is an open set such that $g(c_1) \in W \subset \bigcup \{\tau \mid \tau \in A_g\} - \{f(c_0)\}$. Therefore there exists a neighborhood N'' of 1 such that if $t \in N''$, then $h_t(c_1) \in W$. Let $N = N' \cap N''$. If $t \in N$, then

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in A_g\} - \{f(c_0)\} \neq \varnothing$$

because $[c_0, c_1] \subset [c_0, t_{h_t}]$ and $h_t(c_1) \in W$. Now suppose

$$g(\partial(C-r \cup \{f(c_0)\})) \cap \bigcup \{\tau \mid \tau \in C_g\} \neq \varnothing.$$

Let

$$\varepsilon = d[g(\partial(C-r \cup \{f(c_0)\})) \cap \bigcup \{\tau \mid \tau \in C_g\}, f(c_0)],$$

where d is a metric for X. Then $\varepsilon > 0$, and by an argument similar to the one above, there exists a neighborhood N' of 1 such that if $t \in N'$, then

$$d[h_t(\partial(C-r \cup \{f(c_0)\})) \cap \bigcup \{\tau \mid \tau \in C_g\}, f(c_0)] > \varepsilon/2.$$

Let U' be the $\varepsilon/2$-neighborhood of $f(c_0)$, and let $U = \text{St}(f(c_0), X) \cap U'$. There exists a neighborhood M of c_0 and neighborhood N'' of 1 such that $H(M \times N'') \subset U$. Let $c' \in r(c' \neq c_0)$ such that $[c_0, c'] \subset M \cap [c_0, c_1]$. If $t \in N' \cap N''$, then

$$h_t([c_0, c']) \cap \bigcup \{\tau \mid \tau \in C_g\} = \{f(c_0)\}.$$

Thus if $t \in N' \cap N''$, we may assume that $[c_0, c'] \subset [c_0, t_{h_t}]$. Therefore, by an argument similar to the one above, we can show that there exists a neighborhood N of 1 such that if $t \in N$, then t_{h_t} can be chosen so that

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in A_g\} - \{f(c_0)\} \neq \varnothing.$$

Now we return to the proof of the theorem. There exist 2-simplexes $\tau_1, \tau_2, \ldots, \tau_m$ in X such that $H([c_0, c_1] \times I) \cap \tau_i \neq \varnothing$ for each $i = 1, 2, \ldots, m$, and

$$H([c_0, c_1] \times I) \subset \bigcup_{i=1}^{m} \tau_i.$$

Obviously some subcollection of $\tau_1, \tau_2, \ldots, \tau_m$ is a chain joining s_f and s_g. Suppose that for each such subcollection $\tau_1, \tau_2, \ldots, \tau_n$, $\tau_i \cap \tau_{i+1}$ is a face of a simplex of

C_f for some $i=1, 2, \ldots, n-1$. For each $t \in I$, some subcollection of $\tau_1, \tau_2, \ldots, \tau_m$ is a chain joining s_{h_t} and s_f. Let

$$\Gamma = \{t \mid \text{if } \tau_1, \tau_2, \ldots, \tau_n \text{ is any subcollection of } \tau_1, \tau_2, \ldots, \tau_m \text{ which is a chain joining } s_{h_t} \text{ and } s_f, \text{ then } \tau_i \cap \tau_{i+1} \text{ is a face of some simplex of } C_f \text{ for some } i=1, 2, \ldots, n-1\},$$

and let $t' = \text{glb}\{t \mid t \in \Gamma\}$. Suppose $t'=1$. Observe that if $\rho, \rho' \in A_{h_t}$ for some t, and ρ can be joined to s_f by a subcollection $\tau_1, \tau_2, \ldots, \tau_n$ of $\tau_1, \tau_2, \ldots, \tau_m$ so that $\tau_i \cap \tau_{i+1}$ is not a face of C_f for any $i=1, 2, \ldots, n-1$, then ρ' can be joined to s_f by such a subcollection of $\tau_1, \tau_2, \ldots, \tau_m$. If

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in A_g\} - \{f(c_0)\} \neq \varnothing,$$

then A_{h_t} and A_g have a common simplex and hence each simplex in A_{h_t} can be joined to s_g by a subcollection $\tau_1, \tau_2, \ldots, \tau_n$ of $\tau_1, \tau_2, \ldots, \tau_m$ so that $\tau_i \cap \tau_{i+1}$ is not a face of C_f for any $i=1, 2, \ldots, n-1$. Therefore, if $t'=1$,

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in A_g\} - \{f(c_0)\} = \varnothing$$

for each $t<1$. This contradicts the assertion and hence $t' \neq 1$. By the assertion, there exists a neighborhood N of t' such that if $t \in N$, then t_{h_t} can be chosen so that

$$h_t([c_0, t_{h_t}] \cap [c_0, c_1]) \cap \bigcup \{\tau \mid \tau \in A_{h_{t'}}\} - \{f(c_0)\} \neq \varnothing.$$

If $t \in N$, then each simplex in A_{h_t} can be joined to $s_{h_{t'}}$ by a subcollection $\tau_1, \tau_2, \ldots, \tau_n$ of $\tau_1, \tau_2, \ldots, \tau_m$ so that $\tau_i \cap \tau_{i+1}$ is not a face of C_f for any $i=1, 2, \ldots, n-1$. Since there exist $t \in N \cap \Gamma$, $t' \in \Gamma$. Therefore $t'>0$, and hence there exist $t \in N$ such that $t<t'$. Thus $t' \notin \Gamma$.

The original proof of the following theorem was due to Ross Finney. The author is also indebted to the referee for suggesting a simpler proof.

THEOREM 12. *Let K be a locally finite polyhedron, and let v be a vertex of K. If $h: I \to K$ is a homeomorphism such that $h(0)=v$, then there exists an isotopy $F: I \times I \to K$ such that $F(x, 0)=h(x)$ for all $x \in I$, $F \mid I \times \{1\}$ is a homeomorphism of I onto an edge emanating from v, and $F(0, t)=v$ for all $t \in I$.*

Proof. If $h(I) \not\subset \text{St}(v, K)$, let $\bar{x}$ be the smallest number in I such that $h(\bar{x}) \notin \text{St}(v,K)$. Then $H: I \times I \to K$ defined by $H(x, t)=h(x-tx+tx\bar{x})$ is an isotopy such that $H(x, 0)=h(x)$ for all $x \in I$, $H(x, 1)=h(x\bar{x}) \in [\text{St}(v, K)]^-$ for all $x \in I$, $H(1, 1) \in \partial(\text{St}(v, K))$, and $H(0, t)=v$ for all $t \in I$. If $h(I) \subset \text{St}(v, K)$, then it is easy to see that there is an isotopy $H: I \times I \to K$ such that $H(x, 0)=h(x)$ for all $x \in I$, $H(x, 1) \in [\text{St}(v, K)]^-$ for all $x \in I$, $H(x, 1) \in \partial(\text{St}(v, K))$ if and only if $x=1$, and $H(0, t)=v$ for all $t \in I$. Thus we may assume without loss of generality that $h(I) \subset [\text{St}(v, K)]^-$ and $h(x) \in \partial(\text{St}(v, K))$ if and only if $x=1$. Now define $G: I \times I \to K$ by

$$\begin{aligned} G(x, t) &= xh(1)+(1-x)v, && t \leqq x \leqq 1, \\ &= th(x/t)+(1-t)v, && 0 \leqq x < t. \end{aligned}$$

Then G is an isotopy such that $G \mid I \times \{0\}$ is a homeomorphism of I onto a line segment in $[\mathrm{St}(v, K)]^-$ from v to $h(1)$, $G(x, 1) = h(x)$ for all $x \in I$, and $G(0, t) = v$ for all $t \in I$.

Notation. Let x_0 be a c-point of X, let C_p be a collection of 2-simplexes of X, and let s_p be a c-line of X such that x_0, C_p, and s_p satisfy Definition 1. Let τ be a 2-simplex of C_p, let s_1 and s_2 denote the 1-faces of τ which have x_0 as a vertex, let s_3 denote the 1-face of τ which does not have x_0 as a vertex, and let

$$S = \bigcup \{s \mid s \text{ is a 1-face of a simplex of } C_p,\ x_0 \text{ is not a vertex of } s, \text{ and } s \text{ is not a face of } \tau\}.$$

Using the same notation for the simplexes of C as that used in §3, let $p, p'\colon C \to X$ be the homeomorphisms which satisfy the following properties:

(1) p maps r linearly onto s_p,

(2) p maps r_{1j} linearly onto s_{j-1} for each $j = 2, 3$,

(3) p maps each point of σ_1 into the point of τ which has the same barycentric coordinates,

(4) p maps $r_2 \cup r_3$ linearly onto S,

(5) if L is a line segment from c_0 to $r_2 \cup r_3$, then p maps L linearly onto the line segment from x_0 to $p(L \cap (r_2 \cup r_3))$,

(6) p' maps r linearly onto s_p,

(7) p' maps r_{1j} linearly onto s_{j-1} for each $j = 2, 3$,

(8) p' maps r_1 linearly onto S,

(9) if L is a line segment from c_0 to r_1, then p' maps L linearly onto the line segment from x_0 to $p'(L \cap r_1)$,

(10) p' maps $r_2 \cup r_3$ linearly onto s_3, and

(11) if L is a line segment from c_0 to $r_2 \cup r_3$, then p' maps L linearly onto the line segment from x_0 to $p'(L \cap (r_2 \cup r_3))$.

Note. In the remainder of this paper, when we speak of p and p', we will mean homeomorphisms satisfying the above conditions. This means that $C_p = C_{p'}$ and $s_p = s_{p'}$.

THEOREM 13. *If $f\colon C \to X$ is an imbedding such that $f(c_0)$ is a c-point of X, $C_f = C_p$, and either $s_f = s_p$ or there exists a chain $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes joining s_f and s_p such that $f(c_0)$ is a vertex of τ_i for each i and $\tau_i \cap \tau_{i+1}$ is not a face of a simplex of C_f for any i, then f is isotopic to either p or p′ under an isotopy H such that $H(c_0, t) = f(c_0)$ for each $t \in I$.*

Proof. Since f is continuous, there exists a neighborhood V of c_0 such that $f(V) \subset \mathrm{St}(f(c_0), X)$. Let $c' \in r$ such that $c' \neq c_0$ and $[c_0, c'] \subset V$, and let $c_1 \in [c_0, t_f] \cap [c_0, c']$ such that $c_1 \neq c_0$. There exists $\lambda (0 \leq \lambda < 1)$ such that $c_1 = \lambda c_0 + (1-\lambda)c$. Let $(w, t) \in C \times I$. If $w \in r$, there exists μ $(0 \leq \mu \leq 1)$ such that $w = \mu c_0 + (1-\mu)c$. Define $K\colon C \times I \to X$ by

$$\begin{aligned} K(w, t) &= f(w), \quad \text{if } w \in D, \\ &= f((\mu + t\lambda - t\lambda\mu)c_0 + (1 - \mu - \lambda t + t\lambda\mu)c), \quad \text{if } w \in r. \end{aligned}$$

Then K is an isotopy, $K(w, 0)=f(w)$, and K_1 is an imbedding of C into X such that $K_1(r)\subset(\mathrm{St}(f(c_0), X)-\bigcup\{\tau \mid \tau\in C_f\})\cup\{f(c_0)\}$ and $K_1(w)=f(w)$ for all $w\in D$.

There exists a positive number S such that if

$$D' = \{x \mid x\in\bigcup\{\tau \mid \tau\in C_f\} \text{ and } d(f(c_0), x) < S\},$$

then $D'\subset(\mathrm{int}(f(D)))\cup\bigcup\{\tau \mid \tau\in C_f\}$. Then $f^{-1}(D')\subset\mathrm{int}(D)$. Let A_1 be the annulus bounded by $f^{-1}(\partial D')$ and ∂D, and let k_1 be a homeomorphism of A_1 onto the annulus $\{z \mid 3\leqq|z|\leqq 4\}$ in the plane which sends $f^{-1}(\partial D')$ onto $\{z \mid |z|=3\}$. Let D'' be a disk with center at c_0 such that $D''\subset\mathrm{int}(f^{-1}(D'))$, and let A_2 be the annulus bounded by $\partial D''$ and $f^{-1}(\partial D')$. Let k_2 be a homeomorphism of A_2 onto the annulus $\{z \mid 1\leqq|z|\leqq 2\}$ in the plane which sends $\partial D''$ onto $\{z \mid |z|=1\}$. Define k_3 mapping D'' onto the disk $\{z \mid |z|\leqq 1\}$ in the plane as follows: $k_3(c_0)$ is the origin, $k_3(w)=k_2(w)$ if $w\in\partial D''$, and if L is a line segment from c_0 to $\partial D''$, then k_3 maps L linearly onto the line segment from the origin to $k_3(L\cap\partial D'')$. Then $k_4\colon f^{-1}(D')\to E^2$ defined by $k_4(w)=k_2(w)$, if $w\in f^{-1}(D')-D''$, and $k_4(w)=k_3(w)$, if $w\in D''$, is a homeomorphism of $f^{-1}(D')$ onto the disk $\{z \mid |z|\leqq 2\}$. Define $k_5\colon \{z \mid |z|\leqq 2\}\to\{z \mid |z|\leqq 3\}$ by k_5 of the origin is the origin, $k_5(z)=k_1(k_2^{-1}(z))$, if $|z|=2$, and if L is a line segment from the origin to $\{z \mid |z|=2\}$, then k_5 maps L linearly onto the line segment from the origin to $k_5(L\cap\{z \mid |z|=2\})$. Then $k\colon D\to\{z \mid |z|\leqq 4\}$ defined by $k(z)=k_1(z)$, if $z\in D-f^{-1}(D')$, and $k(z)=k_5k_4(z)$, if $z\in f^{-1}(D')$, is a homeomorphism which sends ∂D onto $\{z \mid |z|=4\}$ and $f^{-1}(\partial D')$ onto $\{z \mid |z|=3\}$ and maps c_0 into the origin. Define $G\colon \{z \mid |z|\leqq 4\}\times I\to\{z \mid |z|\leqq 4\}$ by $G(z, t)=z-tz/4$. Define $F\colon D\times I\to X$ by $F(w, t)=fk^{-1}G(k(w), t)$. Then F is an isotopy, $F(w, 0)=f(w)$, and $F(w, 1)\in D'$. Since $F(c_0, t)=f(c_0)$ for all $t\in I$, we can extend F to an isotopy $F^*\colon C\times I\to X$ by defining $F^*(w, t)=K_1(w)$ for all $w\in r$. Then $F^*(w, 0)=K_1(w)$ for all $w\in C$, and F_1^* is an imbedding of C into X such that $F_1^*(D)=D'$.

Let $w\in D-\{c_0\}$, and let L_1 be the line segment from c_0 to ∂D which passes through w. Then $F_1^*(L_1\cap\partial D)\in\partial D'$. Let L_2 be the line segment from $f(c_0)$ to $\partial(\bigcup\{\tau \mid \tau\in C_f\})$ which passes through $F_1^*(L_1\cap\partial D)$, and let

$$a = L_2\cap\partial(\bigcup\{\tau \mid \tau\in C_f\}).$$

Let e be a metric for C, and let ε be the e radius of D. Define $J\colon C\times I\to X$ by

$$J(w, t) = K_1(w), \text{ if } w\in r,$$

$$J(w, t)=F_1^* \text{ (the point on } L_1 \text{ whose distance from } c_0 \text{ is } 2e(w, c_0)/(2-t)),$$
$$\text{if } w\in D \text{ and } e(w, c_0) \leqq \varepsilon(2-t)/2,$$

and

$$J(w, t) = [(2e(w, c_0)-2\varepsilon+\varepsilon t)/\varepsilon]a+[(3\varepsilon-\varepsilon t-2e(w, c_0))/\varepsilon]F_1^*(L_1\cap\partial D),$$
$$\text{if } w\in D \text{ and } e(w, c_0) \geqq \varepsilon(2-t)/2.$$

Then J is an isotopy, $J(w, 0)=F_1^*(w)$, if $w\in D$, and $J_1(D)=\bigcup\{\tau \mid \tau\in C_f\}$.

It is clear that there exists an isotopy $M^*\colon \partial D\times I \to \partial(\bigcup\{\tau \mid \tau\in C_f\})$ such that $M^*(w,0)=J_1(w)$ and M_1^* is either $p\mid \partial D$ or $p'\mid \partial D$. Also it is clear that this isotopy can be extended to an isotopy $M'\colon D\times I\to \bigcup\{\tau\mid \tau\in C_f\}$ such that $M'_0=J_1$ and $M'(c_0,t)=f(c_0)$. Then we can extend M' to an isotopy

$$M\colon C\times I\to \bigcup\{\tau\mid\tau\in C_f\}\cup f([c_0,c_1])$$

by defining $M(w,t)=K_1(w)$ for all $w\in r$. Now by Alexander's Theorem [1], M'_1 is isotopic to either $p\mid D$ or $p'\mid D$ under an isotopy N' such that $N'(c_0,t)=f(c_0)$. Again N' can be extended to an isotopy $N\colon C\times I\to \bigcup\{\tau\mid\tau\in C_f\}\cup f([c_0,c_1])$ by defining $N(w,t)=K_1(w)$ for all $w\in r$.

The desired result now follows immediately from Theorem 12.

THEOREM 14. *Let $f\colon C\to X$ be an imbedding such that $f(c_0)$ is a c-point of X. If $F\colon C\times I\to X$ is an isotopy such that $F(w,0)=f(w)$ for each $w\in C$ and*

$$t' = \text{glb}\{t\mid F(c_0,t)\neq f(c_0)\},$$

then there exists a neighborhood V of t' such that $F(c_0,t)\in\bigcup\{\tau\mid\tau\in C_f\}$ whenever $t\in V$.

Proof. Suppose that for each neighborhood R of t', there exists $t\in R$ such that $F(c_0,t)\notin\bigcup\{\tau\mid\tau\in C_f\}$. Observe that $F(c_0,t')=f(c_0)$. Let V' be a neighborhood of (c_0,t') such that $F(V')\subset \text{St}(f(c_0),X)$. There exists a connected neighborhood M' of c_0 and a neighborhood N' of t' such that $M'\times N'\subset V'$. Let $c_1\in M'\cap D$ such that $c_1\neq c_0$. Then $F(c_1,t')\in\text{int}(\bigcup\{\tau\mid\tau\in C_f\})$. Let V be any neighborhood of (c_1,t'). There exists a neighborhood M of c_1 and a connected neighborhood N of t' such that $M\times N\subset V\cap(M'\times N')$. There exists $t_1\in N$ such that

$$F(c_0,t_1)\notin\bigcup\{\tau\mid\tau\in C_f\}.$$

Since

$$M'\times\{t_1\}\subset V',\ F(M'\times\{t_1\})\subset \text{St}(f(c_0),X).$$

Let X' be a subdivision of X such that $F(c_0,t_1)$ is a c-point of X', and let $f_{t_1}=F\mid C\times\{t_1\}$. Since $M'\times\{t_1\}$ is connected, $c_0\in M'$, and

$$c_1\in M'\cap D,\ F(c_1,t_1)\in\text{int}(\bigcup\{\tau\mid\tau\in C_{f_{t_1}}\}).$$

Therefore $F(V)\not\subset\text{int}(\bigcup\{\tau\mid\tau\in C_f\})$, and hence F is not continuous.

DEFINITION 4. If $f,g\colon C\to X$ are imbeddings such that $f(c_0)$ and $g(c_0)$ are c-points of X, then we say that $f(C)$ and $g(C)$ are *combinatorially joined* if there exist a sequence $s_1,s_2,\ldots,s_\alpha$ of 1-simplexes and three sequences

$$\tau_1,\tau_2,\ldots,\tau_q;\ \tau'_1,\tau'_2,\ldots,\tau'_m;\ \tau''_1,\tau''_2,\ldots,\tau''_n$$

of 2-simplexes such that:

(1) $f(c_0)$ is a vertex of s_1 and $g(c_0)$ is a vertex of s_α,

(2) $s_\beta\cap s_{\beta+1}$ is a vertex for each $\beta=1,2,\ldots,\alpha-1$,

(3) s_f is a face of τ_1 and s_g is a face of τ_q,

(4) τ'_1 and τ''_1 are simplexes of C_f and τ'_m and τ''_n are simplexes of C_g,

(5) for each i, j, and k, $\tau_i \cap \tau_{i+1}$, $\tau'_j \cap \tau'_{j+1}$, and $\tau''_k \cap \tau''_{k+1}$ are ρ-simplexes ($\rho = 1, 2$), and

(6) for each $\beta = 1, 2, \ldots, \alpha$, we may choose $i(\beta)$, $j(\beta)$, and $k(\beta)$ such that:

(a) $j(1) = 1$, $k(1) = 1$, $j(\alpha) = m$, and $k(\alpha) = n$,

(b) for each $\beta = 1, 2, \ldots, \alpha - 1$, $i(\beta+1) > i(\beta)$, $j(\beta+1) > j(\beta)$, and $k(\beta+1) > k(\beta)$,

(c) $\tau_{i(\beta)}$, $\tau'_{j(\beta)}$, and $\tau''_{k(\beta)}$ are distinct,

(d) $\tau_{i(\beta)} \cap \tau'_{j(\beta)} \cap \tau''_{k(\beta)} = s_\beta$,

(e) if $\tau_{i(\beta)} \cap \tau_{i(\beta+1)}$ is a ρ-simplex ($\rho = 1, 2$), then $\tau_{i(\beta+1)} = \tau_{i(\beta)+1}$,

(f) if $\tau'_{j(\beta)} \cap \tau'_{j(\beta+1)}$ is a ρ-simplex ($\rho = 1, 2$), then $\tau'_{j(\beta+1)} = \tau'_{j(\beta)+1}$,

(g) if $\tau''_{k(\beta)} \cap \tau''_{k(\beta+1)}$ is a ρ-simplex ($\rho = 1, 2$), then $\tau''_{k(\beta+1)} = \tau''_{k(\beta)+1}$,

(h) if $\tau_{i(\beta)} \cap \tau_{i(\beta+1)}$ is a vertex v, then, for each $\gamma = i(\beta)+1, \ldots, i(\beta+1)-1$, each $\delta = j(\beta), \ldots, j(\beta+1)$, and each $\varepsilon = k(\beta), \ldots, k(\beta+1)$, $\tau_\gamma \cap \tau'_\delta = \tau_\gamma \cap \tau''_\varepsilon = \{v\}$,

(i) if $\tau'_{j(\beta)} \cap \tau'_{j(\beta+1)}$ is a vertex v, then, for each $\gamma = i(\beta), \ldots, i(\beta+1)$, each $\delta = j(\beta)+1, \ldots, j(\beta+1)-1$, and each $\varepsilon = k(\beta), \ldots, k(\beta+1)$, $\tau_\gamma \cap \tau'_\delta = \tau'_\delta \cap \tau''_\varepsilon = \{v\}$,

(j) if $\tau''_{k(\beta)} \cap \tau''_{k(\beta+1)}$ is a vertex v, then for each $\gamma = i(\beta), \ldots, i(\beta+1)$, each $\delta = j(\beta), \ldots, j(\beta+1)$, and each $\varepsilon = k(\beta)+1, \ldots, k(\beta+1)-1$, $\tau_\gamma \cap \tau''_\varepsilon = \tau'_\delta \cap \tau''_\varepsilon = \{v\}$,

(k) if $i(1) > 1$, then, for each $i = 1, 2, \ldots, i(1)-1$, $\tau_i \cap \tau'_1 = \tau_i \cap \tau''_1 = \{f(c_0)\}$, and

(l) if $i(\alpha) < q$, then, for each $i = i(\alpha)+1, \ldots, q$, $\tau_i \cap \tau'_m = \tau_i \cap \tau''_n = \{g(c_0)\}$.

We say that $s_1, s_2, \ldots, s_\alpha$ and $\tau_1, \tau_2, \ldots, \tau''_n$ *combinatorially join* $f(C)$ and $g(C)$.

THEOREM 15. *Let f, g: $C \to X$ be imbeddings such that $f(c_0)$ and $g(c_0)$ are c-points of X. If f is isotopic to g under an isotopy H such that $H(c_0, t) \neq f(c_0)$ for some $t \in I$, then $f(C)$ and $g(C)$ are combinatorially joined.*

Proof. We may choose 1-simplexes $s_1, s_2, \ldots, s_\alpha$ in

$$\{s \mid s \text{ is a 1-simplex and } H(\{c_0\} \times I) \cap \operatorname{int}(s) \neq \varnothing\},$$

2-simplexes $\tau_1, \tau_2, \ldots, \tau_q$ in

$$\{\tau \mid \tau \text{ is a 2-simplex and for some } t \in I \text{ arbitrarily small neighborhoods of } H(c_0, t) \text{ intersect } H((r - \{c_0\}) \times \{t\}) \cap \tau\},$$

and 2-simplexes $\tau'_1, \tau'_2, \ldots, \tau'_m$; $\tau''_1, \tau''_2, \ldots, \tau''_n$ in

$$\{\tau \mid \tau \text{ is a 2-simplex and for some } t \in I \text{ arbitrarily small neighborhoods of } H(c_0, t) \text{ intersect } H((D - \{c_0\}) \times (\{t\}) \cap \tau\}$$

so that they may be ordered in such a way as to satisfy Definition 4.

THEOREM 16. *The imbeddings p and p' are not isotopic.*

Proof. Suppose F: $C \times I \to X$ is an isotopy between p and p'. If $F(c_0, t) = x_0$ for all $t \in I$, then $F \mid \partial D \times I$ is an isotopy in $X - \{x_0\}$ between $p \mid \partial D$ and $p' \mid \partial D$, and therefore X is not contractible. Hence there exists $t \in I$ such that $F(c_0, t) \neq x_0$.

Now there exists a sequence $s_1, s_2, \ldots, s_\alpha$ of 1-simplexes and three sequences

$$\tau_1, \tau_2, \ldots, \tau_q;\ \tau_1', \tau_2', \ldots, \tau_m';\ \tau_1'', \tau_2'', \ldots, \tau_n''$$

of 2-simplexes which combinatorially join $p(C)$ and $p'(C)$ and which have the following properties:

(1) $\operatorname{int}(s_\beta) \cap F(\{c_0\} \times I) \neq \varnothing$ for each $\beta = 1, 2, \ldots, \alpha$,

(2) for each i, there exists $t \in I$ such that arbitrarily small neighborhoods of $F(c_0, t)$ intersect $F((r - \{c_0\}) \times \{t\}) \cap \tau_i$,

(3) for each j, there exists $t \in I$ such that arbitrarily small neighborhoods of $F(c_0, t)$ intersect $F((D - \{c_0\}) \times \{t\}) \cap \tau_j'$, and

(4) for each k, there exists $t \in I$ such that arbitrarily small neighborhoods of $F(c_0, t)$ intersect $F((D - \{c_0\}) \times \{t\}) \cap \tau_k''$.

We will assume throughout the remainder of this proof that $F \mid C \times \{0\} = p$ and show that $F \mid C \times \{1\} \neq p'$. First suppose that $p(c_0)$ is not a vertex of s_β for any $\beta = 2, 3, \ldots, \alpha - 1$. If $s_1, s_2, \ldots, s_\alpha$ does not contain a simple closed curve, then it is easy to see that $F \mid C \times \{1\}$ is "essentially" p rather than p' because the isotopy has not "flipped" the disk $\bigcup \{\tau \mid \tau \in C_p\}$. If $s_1, s_2, \ldots, s_\alpha$ contains a simple closed curve, then $F \mid C \times \{1\}$ is "essentially" either p or a rotation of p rather than p' because if the isotopy "flips" the disk $\bigcup \{\tau \mid \tau \in C_p\}$ then $s_p = s_{p'}$ cannot be a face of τ_q. Now if $p(c_0)$ is a vertex of s_β for some $\beta = 2, 3, \ldots, \alpha - 1$, then, in order to determine $F \mid C \times \{1\}$, we examine some finite combination of the possibilities listed above. But it is obvious that this finite combination will "essentially" yield either p or a rotation of p rather than p'. Therefore p is not isotopic to p'.

THEOREM 17. *Let $f, g \colon C \to X$ be imbeddings such that $f(c_0)$ and $g(c_0)$ are c-points of X. If $C_f = C_p$, if either $s_f = s_p$ or there exists a chain $\tau_1, \tau_2, \ldots, \tau_n$ of 2-simplexes joining s_f and s_p such that $f(c_0)$ is a vertex of τ_i for each i and $\tau_i \cap \tau_{i+1}$ is not a face of a simplex of C_f for any i, and if $f(C)$ and $g(C)$ are combinatorially joined, then g is isotopic to either p or p'.*

Proof. By Theorem 13, there is a p_g such that g is isotopic to either p_g or p_g'. It is clear that $p(C)$ and $p_g(C)$ are combinatorially joined. Therefore p_g is isotopic to either p or p', and hence g is isotopic to either p or p'.

THEOREM 18. *If $f \colon C \to X$ is an imbedding and $f(c_0)$ is an interior point of a 1-simplex s of X, then there exists a unique collection D_f consisting of two 2-simplexes of X which contain s as a face such that (1) $f(C - r)$ intersects the interior of every simplex in D_f and (2) there exists a neighborhood U of $f(c_0)$ such that if τ is a simplex which is not a face of a simplex of D_f, then $f(C - r) \cap \operatorname{int}(\tau) \cap U = \varnothing$. Moreover there is a point t_f in r $(t_f \neq c_0)$ and a 2-simplex $\tau \in X - C_f$ such that*

$$f([c_0, t_f] - \{c_0\}) \subset \operatorname{int}(\tau).$$

The proof is essentially the same as the proof of Theorem 10 and hence it is omitted.

Notation. If τ is the 2-simplex such that $f([c_0, t_f]-\{c_0\})\subset \text{int}(\tau)$, let s_f denote the line segment in τ from $f(c_0)$ to the vertex of τ which is not a vertex of s.

Note. Since $f(c_0)$ is a c-point of a subdivision of X, we can obviously define imbeddings p and p' just as before so that $p(c_0)=p'(c_0)=f(c_0)$ and show that f is isotopic to either p or p' but not both.

THEOREM 19. *Suppose f, g: $C\to X$ are imbeddings such that $f(c_0)$ and $g(c_0)$ are interior points of 1-simplexes s_1 and s_2 respectively ($s_1\neq s_2$). Then f is isotopic to g if and only if there exist imbeddings h, k: $C\to X$ such that $h(c_0)$ and $k(c_0)$ are c-points, f is isotopic to h, g is isotopic to k, and h is isotopic to k.*

Proof. Suppose F: $C\times I\to X$ is an isotopy such that $F(w, 0)=f(w)$ and $F(w, 1)=g(w)$ for all $w\in C$. Suppose that $F(c_0, t)$ is not a c-point of X for any $t\in I$. If $t_1=\text{lub}\{t \mid F(c_0, t)\in s_1\}$, then F is not continuous at (c_0, t_1).

If the condition is satisfied, then f is isotopic to g because isotopy is an equivalence relation.

THEOREM 20. *Suppose f: $C\to X$ is an imbedding such that $f(c_0)$ is an interior point of a 1-simplex s of X. Then there exists an imbedding g: $C\to X$ such that $g(c_0)$ is a c-point of X and f is isotopic to g if and only if there exists a vertex v of s, 2-simplexes $\tau_1, \tau_2, \ldots, \tau_n$ of X, and a 1-simplex s_1 of X such that:*

(1) $v, \tau_1, \tau_2, \ldots, \tau_n$, and s_1 satisfy Definition 1,

(2) $\bigcup\{\tau \mid \tau\in D_f\}\subset \bigcup_{i=1}^n \tau_i$, and

(3) either s_1 and s_f are in the same 2-simplex or there exists a chain $\tau_1', \tau_2', \ldots, \tau_q'$ of 2-simplexes such that $s_1\subset\tau_1'$, $s_f\subset\tau_q'$, v is a vertex of τ_j' for each j, and $\tau_j'\cap\tau_{j+1}'$ is a 1-simplex which is not a face of τ_i for any i.

Proof. Suppose there exists an imbedding g: $C\to X$ such that $g(c_0)$ is a c-point of X and f is isotopic to g. Let F: $C\times I\to X$ be an isotopy such that $F(w, 0)=f(w)$ and $F(w, 1)=g(w)$ for all $w\in C$. Let $t_1=\text{glb}\{t\in I \mid F(c_0, t) \text{ is a vertex of } s\}$. Then $F(c_0, t_1)$ is a vertex v of s, and it is clear that the imbedding f_{t_1}: $C\to X$ defined by $f_{t_1}(w)=F(w, t_1)$ gives us a collection of simplexes satisfying the condition.

Suppose the condition is satisfied. By the note preceding Theorem 19, there is a p such that f is isotopic if either p or p'. It is clear that p, and hence p', is isotopic to an imbedding g: $C\to X$ such that $g(c_0)$ is a c-point of X.

THEOREM 21. *Let s be a 1-simplex of X which does not have a c-point as vertex but which is a face of at least three 2-simplexes. If n is the number of 2-simplexes which have s as a face, then $\{f$: $C\to X \mid f$ is an imbedding and $f(c_0)$ is an interior point of $s\}$ consists of $6C(n, 3)$ isotopy classes.*

Proof. It is clear that if either $D_f\neq D_g$ or $f([c_0, t_f])$ and $g([c_0, t_g])$ are in different simplexes, then f is not isotopic to g. Thus the theorem follows since there exists p such that if $D_f=D_p$ and $f([c_0, t_f])$ and $p(r)$ are in the same simplex, then f is isotopic to either p or p' but not both.

Summary. Now it follows that in order to compute the number of isotopy classes of imbeddings of C in X, it is sufficient to consider only the c-points and the 1-simplexes which do not have a c-point as vertex but which are faces of at least three 2-simplexes. Let $x_1, x_2, \ldots, x_m$ denote the c-points of X, and let $s_1, s_2, \ldots, s_n$ denote the 1-simplexes which do not have a c-point as vertex but which are faces of at least three 2-simplexes. For each $i=1, 2, \ldots, m$, let $C_{i1}, C_{i2}, \ldots, C_{iq_i}$ be the collections of 2-simplexes having x_i as a vertex and satisfying Definition 1. Suppose $1 \leqq i \leqq m$ and $1 \leqq k \leqq q_i$. For each 2-simplex τ such that x_i is a vertex of τ and the 1-faces of τ which have x_i as a vertex are faces of simplexes of C_{ik}, choose a line segment in τ from x_i to the barycenter of the 1-face of τ opposite x_i, and let s_{ik1}, $s_{ik2}, \ldots, s_{ik\alpha_{ik}}$ denote this collection of line segments together with all 1-simplexes having x_i as a vertex which are not faces of simplexes of C_{ik}. Corresponding to (C_{11}, s_{111}), there are 2 isotopy classes of imbeddings of C in X. Corresponding to (C_{11}, s_{112}), there are 2 isotopy classes of imbeddings of C in X. We examine these to see if either is one of the 2 classes previously obtained. We will either get 2 new classes or no new classes. We continue this process. For each $(C_{ik}, s_{ik\beta})$, $i=1, 2, \ldots, m$; $k=1, 2, \ldots, q_i$; $\beta=1, 2, \ldots, \alpha_{ik}$, there are 2 isotopy classes of imbeddings of C in X. They are either both new or neither is new. Let γ_1 be the number of distinct isotopy classes of imbeddings of C in X obtained from $(C_{1k}, s_{1k\beta})$, $k=1, 2, \ldots, q_1$; $\beta=1, 2, \ldots, \alpha_{1k}$. For each $i=2, 3, \ldots, m$, let γ_i be the number of distinct isotopy classes of imbeddings of C in X obtained from $(C_{ik}, s_{ik\beta})$, $k=1, 2, \ldots, q_i$; $\beta=1, 2, \ldots, \alpha_{ik}$, which are different from those obtained from $(C_{ak}, s_{ak\beta})$, $a=1, 2, \ldots, i-1$; $k=1, 2, \ldots, q_a$; $\beta=1, 2, \ldots, \alpha_{ak}$. For each $j=1, 2, \ldots, n$, let n_j be the number of 2-simplexes which have s_j as a face. Then the number of isotopy classes of imbeddings of C in X is

$$\sum_{i=1}^{m} \gamma_i + 6 \sum_{j=1}^{n} C(n_j, 3).$$

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